Hysteresis in Ferromagnetic Random Field Ising Model with an Arbitrary Initial State

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We present exact expressions for hysteresis loops in the ferromagnetic random field Ising model in the limit of zero temperature and zero driving frequency for an arbitrary initial state of the model on a Bethe lattice. This work extends earlier results that were restricted to an initial state with all spins pointing parallel to each other.

I. INTRODUCTION

Zero temperature Glauber dynamics of a driven ferromagnetic random field Ising model (RFIM) provides an interesting caricature of hysteresis, Barkhausen noise, and return point memory in several complex systems [1]. It also provides a simple example of non-equilibrium critical behavior along with scaling and universality commonly associated with equilibrium critical phenomena. In view of these features, the model has been studied extensively by numerical simulations on hypercubic lattices, and by exact solution in one dimension, and on a Bethe lattice in special cases [2,3,4,5,6]. Results on the Bethe lattice show an unexpected dependence of non-equilibrium critical behavior on the coordination number of the lattice. The non-equilibrium critical point occurs only if $z > 3$, where z is the coordination number of the lattice. These and other results have been obtained in the limit of zero temperature and zero driving frequency, starting from a state of saturated magnetization, i.e. when all spins are initially pointing parallel to each other. In the present paper, we extend the earlier results by obtaining an analytic expression for the hysteresis loop starting from an arbitrary initial state.

II. THE MODEL

Ising spins S_i ($S_i = \pm 1$) are located on a lattice at sites labeled by an integer i = 1 to N. Nearest neighbor spins interact via a ferromagnetic interaction J (J>0). A site-dependent quenched random field h_i acts on S_i in addition to a uniform external field h that acts on all spins. The fields $\{h_i\}$ are independent identically distributed random variables drawn from a continuous probability distribution $\phi(h_i)$. The Hamiltonian of the system is

$$H = -J\sum_{i,j} S_i S_j - \sum_i h_i S_i - h\sum_i S_i = -\sum_i f_i S_i, \tag{1}$$

where f_i is the net field acting on the spin S_i: $f_i = J\sum_j S_j + h_i + h$. The system evolves through single-spin-flip Glauber dynamics at zero temperature that lowers the energy of the system by aligning spins along the net field at their respective sites, i.e. S_i is flipped if it does not have the same sign as f_i. The dynamics is an iterative process because flipping of a spin alters the net field on its neighbors, and may necessitate the flipping of neighbors and their neighbors. In practice, the dynamics is implemented by the following rule: choose a spin at random, and flip it only if it is not aligned along the net field at its site. Repeat the process till all spins are aligned along the net fields at their respective sites.

III. HYSTERESIS ON A BETHE LATTICE

A Bethe lattice is an infinite-size branching tree of coordination number z. We consider a random configuration of spins on this lattice such that the spin on any given site is up with probability f and down with probability $1-f$. Two special values of f equal to zero and unity that correspond to an initial state with all spins parallel to each other have been examined earlier in the study of hysteresis on a Bethe lattice. The purpose of the present paper is to extend the analytic results obtained earlier to all values of f in the range $0 \le f \le 1$. As the external field is slowly ramped up from $-\infty$ to ∞, we ask what fraction of spins are up at an intermediate applied field h. We choose a site at random in the deep interior of the tree, and designate it as the central site. We need to calculate the probability $p(h)$ that the central site is up when the applied field has been increased from $-\infty$ to h. The fraction f sets the lower bound on $p(h)$ because spins that are up in the initial state must remain up in an increasing field. Indeed, the fraction f of

spins that are up in the initial state may be considered "quenched" because they do not form dynamical variables of the problem in an increasing applied field. Thus the problem reduces to finding the probability that the central site is up given that it was down initially. In the following, we calculate the complementary probability $q(h) = 1 - p(h)$ that the central site is down when the system is relaxed at field h. The magnetization per site is given by $m(h) = 1 - 2q(h)$.

Each nearest neighbor of the central site forms the vertex of an infinite sub-tree. The sub-trees do not interact with each other except through the central site. Therefore the evolution on z sub-trees meeting at the central site is independent of each other as long as the central site does not flip up from its initial state. In ferromagnetic dynamics the same final stable state is obtained irrespective of the order in which we relax the central site and its nearest neighbors. This is known as the Abelian property of the ferromagnetic dynamics. We choose to relax the central site after its neighbors have been relaxed. Let $Q^*(h)$ denote the conditional probability that a nearest neighbor of the central site is down after it has been relaxed at field h given that the central site is is held down at h. The central site is relaxed after its nearest neighbors have been relaxed. The probability that the central site stays down after it is allowed to relax is given by,

$$q(h) = (1-f)\sum_{m=0}^{z}\left(\begin{array}{c} z \\ m \end{array}\right)[1-Q^*(h)]^m[Q^*(h)]^{z-m}[1-p_m(h)] \tag{2}$$

The first factor $(1-f)$ gives the probability that the central site is initially down, the last factor, $[1-p_m(h)]$, gives the probability that a spin with m nearest neighbors up, and $z-m$ neighbors down does not have sufficient random quenched field at its site to flip it up at an applied field h.

$$p_m(h) = \int_{(z-2m)J-h}^{\infty}\phi(h_i)dh_i \qquad \text{(m=0,...,z)} \tag{3}$$

The conditional probability $Q^*(h)$ that enters equation (2) is given by the fixed point of the following recursion relation.

$$Q^* = \lim_{n\to\infty} Q^n;\; Q^n = (1-f)\sum_{m=0}^{z-1}\left(\begin{array}{c} z-1 \\ m \end{array}\right)[1-Q^{n-1}]^m[Q^{n-1}]^{z-1-m}(1-p_m) \tag{4}$$

The rationale for the recursion relation (4) is as follows. Each nearest neighbor of the central site forms the vertex of a sub-tree. Let the central site and one of its nearest neighbors be at a distance of $n+1$ and n steps respectively from the boundary of the corresponding sub-tree. Q^n denotes the conditional probability that the vertex is down when relaxed at field h given that the central site is down before the vertex is relaxed. In our scheme, the sites at the boundary of the sub-tree are relaxed first, then the sites located one step above the boundary and so on. The recursion relation works one step at a time from the boundary to the vertex. Only non-quenched sites are relaxed. The probability that the vertex is not quenched is equal to $1-f$. This accounts for the first factor on the right side of the recursion relation. Given that the central site is down, the vertex has $z-1$ neighbors at height $n-1$ that than be independently down or up as a result of the branching structure of the tree. Q^{n-1} is the conditional probability that any one of the $(z-1)$ neighbors at the lower level is down when relaxed at h given that the vertex is down; $[1-Q^{n-1}]$ is the probability that a neighbor at the lower level is up. The up neighbor may be a quenched or a non-quenched site. The three factors following the summation sign in equation (4) give the probability that m out of $(z-1)$ sites are up, and the rest are down. The last factor $(1-p_m)$ gives the probability that the vertex remains down when relaxed. The recursion relation (4) is similar to the one obtained earlier in the case when the initial state has all spins pointing parallel to it, and reduces to the earlier result if $f=0$, as it should. However, for $0 \leq f \leq 1$ the lattice is punctuated by quenched sites, and we relax only the non-quenched sites. The relaxed state of the vertex is determined by the longest path of "non-quenched" sites connected to it. The longest possible path is the one that connects the boundary to the vertex. As, $n \to \infty$, the recursion relation iterates to the fixed point solution $Q^*(h)$ which incorporates in it all shorter paths with suitable weights.

If the applied field is reversed before completing the lower half of the major loop, a minor hysteresis loop is generated. First reversal of the field generates the upper half of the minor loop, and a second reversal generates the lower half. When the field on second reversal reaches the point where the first reversal was made, the lower half of the minor loop meets the starting point of the upper half. In other words, the minor loop closes upon itself at the point it started. This property of the RFIM is known as return point memory. The analytic calculation of minor loop is

somewhat more tedious than that of the major loop. Consider the upper half of the minor loop. Suppose the applied field is reversed from h to h' ($h' \leq h$). By assumption, the quenched sites do not turn down in the reversed field. We need to calculate the probability that a non-quenched site that was up at h turns down at h'. When a non-quenched site i turns up, the field on its nearest neighbors increases by an amount $2J$. The increased field may cause some non-quenched neighbors to turn up. Each neighbor that turns up after site i, increases the field on site i by an amount $2J$. Therefore, in the reversing field site i can turn down only after all neighbors which turned up after it have turned down. The probability $D^*(h')$ that a non-quenched nearest neighbor of site i that was down before site i turned up is again down at h' is determined by the fixed point of the following recursion relation,

$$D^n(h') = (1-f)\sum_{m=0}^{z-1}\binom{z-1}{m}[1-Q^*(h)]^m[Q^*(h)]^{z-1-m}\left[1-p_{m+1}(h)\right]$$
$$+(1-f)\sum_{m=0}^{z-1}\binom{z-1}{m}[1-Q^*(h)]^m[D^{n-1}(h')]^{z-1-m}\left[p_{m+1}(h)-p_{m+1}(h')\right] \tag{5}$$

Given a site i that is up at h, the first sum above gives the conditional probability that a non-quenched nearest neighbor of site i remains down at h after site i has turned up. The second sum takes into account the situation that the nearest neighbor in question turns up at h after site i turns up but turns down at h'.

The fraction of non-quenched sites that turn down at h' is given by,

$$q'(h') = (1-f)\sum_{m=0}^{z}\binom{z}{m}[1-Q^*(h)]^m[D^*(h')]^{z-m}\left[p_m(h)-p_m(h')\right] \tag{6}$$

Consequently, for $h-2J \leq h' \leq h$, the magnetization on the upper return loop is given by,

$$m'(h') = 1-2[q(h)+q'(h')] \tag{7}$$

At $h' = h-2J$, all neighbors of the central site that flipped up because the central site flipped up at h have flipped down, and the neighborhood of the central site is stable with the central site in the up position. As the applied field decreases further, the central site must turn down before any of its nearest neighbors. This means that at field $h-2J$ the system arrives at some point on the upper half of the major hysteresis loop, and moves on it upon further decrease in the applied field. Thus the magnetization for $h' < h-2J$ is given by the formula,

$$m'(h') = 1-2\tilde{q}(h'), \tag{8}$$

where,

$$\tilde{q}(h') = (1-f)\sum_{m=0}^{z}\binom{z}{m}[1-\tilde{Q}^*(h')]^m[\tilde{Q}^*(h')]^{z-m}[1-p_m(h')], \tag{9}$$

and $\tilde{Q}^*(h')$ is given by the fixed point of the following recursion relation.

$$\tilde{Q}^n = (1-f)\sum_{m=0}^{z-1}\binom{z-1}{m}[1-\tilde{Q}^{n-1}]^m[\tilde{Q}^{n-1}]^{z-1-m}(1-p_{m+1}) \tag{10}$$

The lower half of the return loop is obtained by reversing the field from h' to h'' ($h'' > h'$). If $h' < h-2J$, the lower half of the minor loop starts from the major loop, and therefore it is related by symmetry to the upper half of the return loop that has been obtained above. We only need to consider the case $h' \geq h-2J$. In this case, the magnetization on the lower half of the return loop may be written as,

$$m''(h'') = 1-2[q(h)+q'(h')-p''(h'')], \tag{11}$$

where $p''(h'')$ is the probability that a non-quenched site that is up at field h and down at field h', turns up again at h''.

$$p''(h'') = (1-f)\sum_{m=0}^{z}\begin{pmatrix} z \\ m \end{pmatrix}[U^*(h'')]^m[D^*(h')]^{z-m}\left[p_m(h'') - p_m(h')\right] \tag{12}$$

Here $U^*(h'')$ is the conditional probability that a nearest neighbor of a site i turns up before site i turns up on the lower return loop. It is determined by the fixed point of the equation,

$$U^n(h'') = 1 - Q^*(h) - (1-f)\sum_{m=0}^{z-1}\begin{pmatrix} z-1 \\ m \end{pmatrix}[1-Q^*(h)]^m[D^*(h')]^{z-1-m}[p_m(h) - p_m(h')]$$
$$+(1-f)\sum_{m=0}^{z-1}\begin{pmatrix} z-1 \\ m \end{pmatrix}[U^{n-1}(h'')]^m[D^*(h')]^{z-1-m}\left[p_m(h'') - p_m(h')\right] \tag{13}$$

The rationale behind equation (13) is similar to the one behind equation (5). Given that a site i is down at h', the first three terms on the right hand side account for the probability that a nearest neighbor of site i is up at $h'' \geq h'$. Note that the neighbor in question must have been up at h in order to be up at h', and if it it is already up at h' then it will remain up on the entire lower half of the return loop, i.e. at $h'' \geq h'$. The last term gives the probability that the neighboring site was down at h', but turned up on the lower return loop before site i turned up. It can be verified that the lower return loop meets the lower major loop at $h'' = h$ and merges with it for $h'' > h$, thus proving the property of return point memory.

The method of calculating the minor loop may be extended to obtain a series of smaller minor loops nested within the minor loop obtained above. The key point is that whenever the applied field is reversed, a site i may flip only after all neighbors of site i which flipped in response to the flipping of site i on the immediately preceding sector have flipped back. The neighbors of site i which did not flip on the preceding sector in response to the flipping of site i do not flip in the reversed field before site i has flipped. We have obtained above expression for the return loop when the applied field is reversed from $h_{ext} = h$ on the lower major loop to $h_{ext} = h'$ ($h - 2J \leq h' \leq h$), and reversed again from $h_{ext} = h'$ to $h_{ext} = h''$ ($h'' \leq h$). When the applied field is reversed a third time from h'' to h''' ($h''' < h''$), expressions for the magnetization on the nested return loop follow the same structure as the one on the trajectory from h to h'. Qualitatively, the role of P^* on the first leg (h to h') is taken up by U^* on the third leg (h'' to h''') of the nested return loop.

IV. COMPARISON OF THEORY WITH SIMULATIONS

A branching tree has large surface effects. Special care has to be taken to eliminate surface effects in theory and simulations before the two can be compared. We eliminate surface effects somewhat differently in simulations than in the theoretical formalism. The analytic results on the Bethe lattice are obtained by taking the infinite-size limit of a branching tree. Fixed points of recursion relations have the effect of eliminating surface effects. It is rather difficult to eliminate surface effects in numerical calculations on branching trees because most of the sites on a finite tree are on the boundary or close to the boundary. We therefore perform the numerical simulations of the model on random graphs of coordination z. A random graph of N sites has no surface, but the price we pay is that it has some loops. However, for almost all sites in the graph, the local connectivity up to a distance of $\log_{(z-1)}$ N is similar to the one in the deep interior of the branching tree. Therefore, simulation on a random graph is a very efficient method of subtracting the surface effects on the corresponding finite branching tree.

We choose to work with a Gaussian distribution of the random field with mean value zero, and variance σ^2 for our numerical work. Our theoretical expressions are valid for any continuous distribution of the random field. Figure (1) shows a comparison between theory, and simulation for z=3, $f = .2$, and $\sigma = 1$. A major loop as well as two minor loops are shown; the minor loops are obtained by reversing the field at $h_1 = .6$ (J=1) on the lower half of the major loop , and making round trip excursions upto $h_2 = -1.4$, and $h_2 = -1$ respectively. Results from simulations have been superposed on the theoretical curves and are indistinguishable from it on the scale of the graph.

Hysteresis on a Bethe lattice of coordination $z \geq 4$ is qualitatively different from the case z=2, and z=3. For $z \geq 4$, there exists a pair of values of σ and h ($\sigma = \sigma_c$, $h = h_c$) that marks a critical point in the response of the system to the driving field. This critical point does not exist on lattices with $z = 2$, and 3. The critical point is marked by the disappearance of a first order jump in the magnetization in an increasing or decreasing applied field. If $f = 0$ and $\sigma < \sigma_c$, the jump in the magnetization occurs at an applied field $h_J >$ J. As σ increases to σ_c, h_J decreases to J, and

the size of the jump reduces to zero. Thus in the case $f = 0$, $h_c = J$ independent of the value of z as long as $z \geq 4$. For $z = 4$, $\sigma_c = 1.78$ approximately. The value of σ_c increases with increasing z. These features remain qualitatively true for the enire range $0 \leq f \leq 1$. When a fraction of spins are quenched, the system is effectively more disordered and it takes relatively smaller disorder in the quenched fields to produce the same effect as in a system with $f = 0$. For example, if $f = .2$ the value of σ_c is reduced from 1.78 to 1.2 approximately. Figure (2) shows the lower halves of the major hysteresis loop for $z = 4$, $f = .2$, and $\sigma = 1.0, 1.2$, and 1.4 respectively. The graph corresponding to $\sigma = 1.4$ is smooth, and the one for $\sigma = 1$ has a first order jump at $h = .334$ approximately. Results of a simulation have been superimposed on the theoretical graphs. The agreement is quite satisfactory within numerical errors.

Figure (3) shows the graph for $z = 4$, $f = .2$, and $\sigma = 1.0$ in detail. The magnetization in increasing field jumps from $m \approx .5$ to $m \approx .97$ at $h_2 \approx .334$. At an earlier value of the applied field $h_1 < h_2$ ($h_1 \approx .31$), the line of fixed points of equation (4) splits into three branches but only one of these branches is stable in increasing field. The system makes a transition from one stable branch to another stable branch such that the magnetization varies smoothly across the field h_1. If one starts with an initial value $Q^0 = 0$ and iterates recursion relation (4) to get the fixed point solution Q^*, then Q^* jumps at $h_1 \approx .31$ to a new but unstable fixed point. The corresponding magnetization jumps from $m \approx .25$ to $m \approx .9$ at $h_1 \approx .31$. This is indicated in figure (3) by a vertical line. However, the higher value of magnetization is unstable and it is not observed in simulations. In simulations the magnetization continues smoothly from h_1 to h_2 ($h_2 > h_1$), and jumps up at h_2 as indicated in the figure by a broken vertical line. The magnetization on the lower hysteresis loop from h_1 to h_2 corresponds to a stable fixed point solution of equation (4) with the initial condition $Q^0 = 1$. In the range $h_1 \leq h \leq h_2$, the fixed point equation (4) has three solutions. The magnetizations obtained by substituting the three fixed point solutions in equation (2) are shown by an s-shaped curve in figure (3). The middle portion of the s-shaped curve showing decreasing magnetization in increasing field is unphysical and is not observed in simulations. The fields h_1 and h_2 mark the the two turning points of the s-shaped curve. With increasing σ the unstable segment shrinks, and the s-shaped graph straightens out. At the critical value σ_c, the magnetization corresponds to a double root of the fixed point equations.

V. CONCLUDING REMARKS

We have obtained analytic expressions for hysteresis loops in the ferromagnetic random field Ising model on a Bethe lattice in the limit of zero temperature and zero driving frequency. Our results are applicable to an arbitrary initial state of the system, and constitute a generalization of results that were restricted to an initial state with all spins pointing parallel to each other. We have checked the analytic results against simulations of the model on random graphs and found the agreement between theory and simulation to be quite satisfactory. It is remarkable that a relatively minor modification of the recurrence relations allows us to pass from an initial state with all spins parallel to each other to a random initial configuration. This apparent simplicity results from the use of fixed points of recurrence relations for conditional probabilities. Let us consider the fixed point solution Q^* for $z = 2$:

$$Q^* = \left[\frac{(1-p_1)}{1-(1-f)(p_1-p_0)}\right] = (1-p_1)\sum_{m=0}^{\infty}[(1-f)(p_1-p_0)]^m \qquad (z=2) \tag{14}$$

Imagine a chain of Ising spins with a fraction f quenched in the up position, and the rest down at $h = -\infty$. Equation (14) indicates that given a down spin say at site i, the conditional probability that its nearest neighbor, say on the right side at site $i+1$ is down when relaxed at field h is equal to the probability that the closest avalanche on the right side comes to a stop after m steps ($m = 1, 2, \ldots$). The prefactor $(1-p_1)$ gives the probability that the avalanche has stopped, and the factor after the summation sign gives the probability of an avalanche of size m. The above argument is not entirely transparent. We have also obtained hysteresis curves for $z = 2$ for $f = 0$ as well as for other values of f by an alternate and more tedious method that considers various probabilities of initiating an avalanche and calculating its resulting size. The alternate method gives the same result as the fixed point method. It shows that a fixed point description of the system contains information on all fluctuations of the system with appropriate weights.

Finally, the results presented here can be extended to systems with impurities and vacancies. In our formalism a fraction f of sites are quenched in the up position and contribute a constant magnetization (per site) equal to $2f-1$. If the constant term is subtracted, and the number of up nearest neighbors of a site is suitably reduced before that site is relaxed, the results can be extended to hysteresis in systems that have quenched random fields, as well as a

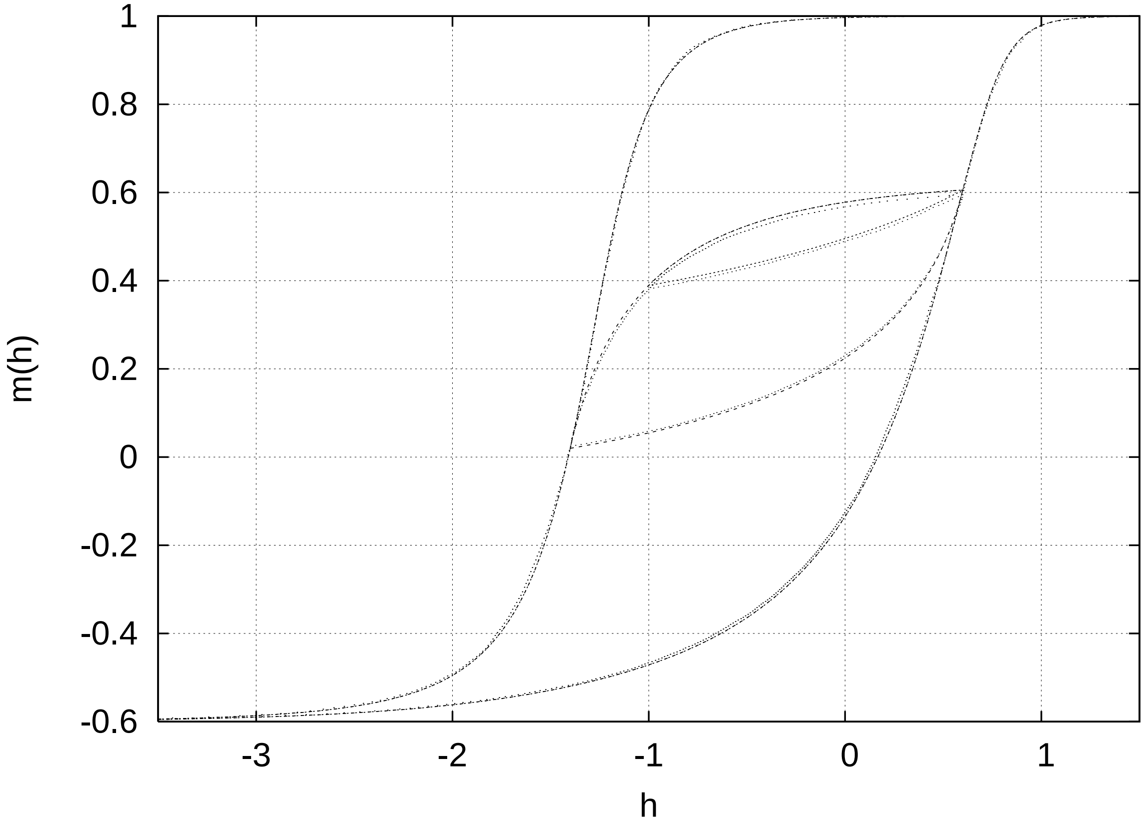

FIG. 1: Hysteresis on a Bethe lattice with $z = 3$, $f = .2$, and $\sigma = 1$. Continuous lines show the theoretical result, and dots mark the results of numerical simulations. A major hysteresis loop and two minor loops are shown; the minor loops are obtained by reversing the field at $h_1 = .6$ (J=1) on the lower half of the major loop , and making round trip excursions upto $h_2 = -1.4$, and $h_2 = -1$ respectively. Theory and simulations are indistinguishable from each other on the scale of the graph.

fraction f of magnetic ions missing.

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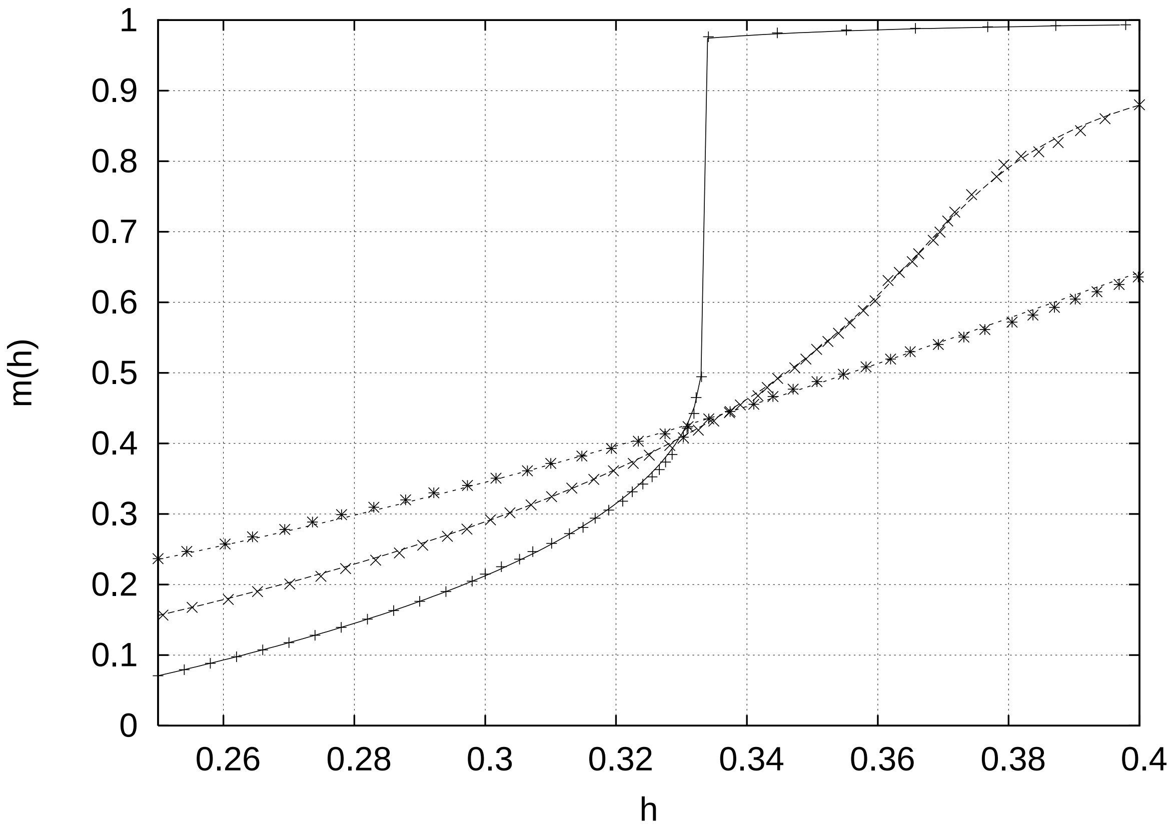

FIG. 2: Magnetization on a Bethe lattice with $z = 4$, and $f = .2$ in an increasing field. Continuous lines show theoretical results for $\sigma = 1.0$J, $\sigma = 1.2$J, and $\sigma = 1.4$J . Results from simulations are superimposed on the corresponding theoretical curves.

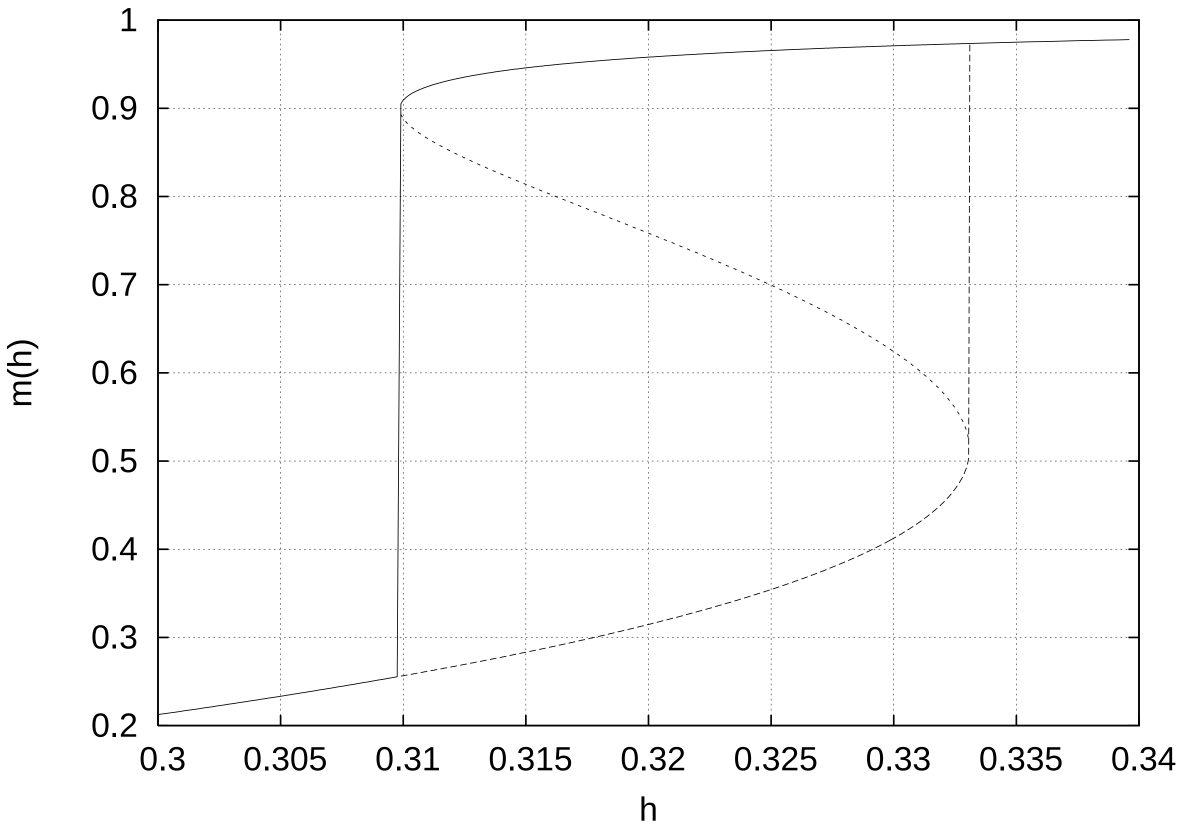

FIG. 3: The s-shaped magnetization curve ($z = 4$, $f = .2$, $\sigma = 1.0$J) showing three fixed point solutions of recursion relations. See text for details.